An attempt to classification of the quasi rational solutions to the NLS equation

Pierre Gaillard

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P. Gaillard,
Université de Bourgogne, Institut de Mathématiques de Bourgogne
Faculté des Sciences Mirande,
9 avenue Alain Savary BP 47870
21078 Dijon Cedex, France :
E-mail : Pierre.Gaillard@u-bourgogne.fr

December 12, 2015

Abstract

Based on a representation in terms of determinants of order $2N$, an attempt to classification of quasi rational solutions to the one dimensional focusing nonlinear Schrödinger equation (NLS) is given and several conjectures about the structure of the solutions are also formulated. These solutions can be written as a product of an exponential depending on t by a quotient of two polynomials of degree $N(N+1)$ in x and t depending on $2N-2$ parameters. It is remarkable to mention that in this representation, when all parameters are equal to 0, we recover the P_N breathers.

PACS numbers :
33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

1 Introduction

The term of rogue wave was introduced in the scientific community by Draper in 1964 [1]. The usual criterion to describe rogue waves in the ocean can be formulated as follows : the vertical distance from trough to crest is two or more times greater than the average wave height among one third of the highest waves in a time series (10 to 30 min). Such types of waves have already been observed; the first rogue wave was recorded by scientific measurement in North Sea was made on the oil platform of Draupner in 1995.
We consider the one dimensional focusing nonlinear Schrödinger equation (NLS) to describe the phenomenon of the rogue waves. This equation was first solved by Zakharov and Shabat in 1972 by using the inverse scattering method [2, 3]. The first quasi rational solutions to NLS equation were constructed in 1983 by Peregrine [4]. It is well known that Akhmediev, Eleonski and Kulagin obtained the first higher order analogue of the Peregrine breather [5, 6] in 1986, and other analogues of the Peregrine breathers of order 3 and 4 were constructed in a series of articles by Akhmediev et al. [7, 8, 9] using Darboux transformations. Since the beginning of the years 2010, many works about NLS equation have

been published. Rational solutions to the NLS equation were written in 2010 as a quotient of two wronskians [10]; in [11] another representation of the solutions to the NLS equation in terms of a ratio of two wronskians of even order $2N$ using truncated Riemann theta functions in 2011 was given; in 2012, Guo, Ling and Liu found solutions as a ratio of two determinants [12] using generalized Darboux transformation; a new approach was proposed by Ohta and Yang in [13] using Hirota bilinear method; in 2013 rational solutions in terms of determinants which do not involve limits were given in [14].
A new representation has been found as a ratio of a determinant of order $N+1$ by another one of order N by Ling and Zhao in [15]. Very recently in 2014, another approach has been given in [16] using a dressing method in which the solutions are expressed as the quotient of a determinant of order $N+1$ by another one of order N.
Here we present multi-parametric families of quasi rational solutions to NLS equation of order N in terms of determinants of order $2N$ depending on $2N-2$ real parameters. With this representation, at the same time, the well-known ring structure, but also the triangular shapes also given by Ohta and Yang [13], Akhmediev et al. [17] are found.
These solutions can be expressed as a ratio of two polynomials of degree $N(N+1)$ of x and t multiplied by an exponential depending on t. It is important to stress that with this representation we get Peregrine breathers P_N of order N when all parameters are equal to 0.
We summarize our results obtained on solutions to NLS equation. Based on this study, we can deduce a classification of the solutions to the present NLS equation. For a generalized NLS equation with complementary terms of nonlinearity of order 5, a classification have already been done by Winternitz and Gagnon using an algebraic approach [18, 19, 20]. In no case here we use such an approach. We try to classify the quasi rational solutions by means of the patterns of their modulus in the (x,t) plane according to the order N and the parameters $\tilde{a}_j$ and $\tilde{b}_j$, $1 \leq j \leq N-1$.

2 Families of solutions to NLS equation depending on $2N-2$ parameters.

We consider the focusing NLS equation

$$iv_t + v_{xx} + 2|v|^2 v = 0. \tag{1}$$

Then we get the following result [21] :

Theorem 2.1. *Function v defined by*

$$v(x,t) = \exp(2it - i\varphi) \times \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} \tag{2}$$

is a quasi-rational solution to the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2 v = 0,$$

quotient of two polynomials $N(x,t)$ and $D(x,t)$ depending on $2N-2$ real parameters $\tilde{a_j}$ and $\tilde{b_j}$, $1 \leq j \leq N-1$.
N and D are polynomials of degrees $N(N+1)$ in x and t, where

$$\begin{array}{l}
n_{j1} = \varphi_{j,1}(x,t,0),\ 1 \leq j \leq 2N \quad n_{jk} = \frac{\partial^{2k-2}\varphi_{j,1}}{\partial\epsilon^{2k-2}}(x,t,0), \\
n_{jN+1} = \varphi_{j,N+1}(x,t,0),\ 1 \leq j \leq 2N \quad n_{jN+k} = \frac{\partial^{2k-2}\varphi_{j,N+1}}{\partial\epsilon^{2k-2}}(x,t,0), \\
d_{j1} = \psi_{j,1}(x,t,0),\ 1 \leq j \leq 2N \quad d_{jk} = \frac{\partial^{2k-2}\psi_{j,1}}{\partial\epsilon^{2k-2}}(x,t,0), \\
d_{jN+1} = \psi_{j,N+1}(x,t,0),\ 1 \leq j \leq 2N \quad d_{jN+k} = \frac{\partial^{2k-2}\psi_{j,N+1}}{\partial\epsilon^{2k-2}}(x,t,0), \\
2 \leq k \leq N,\ 1 \leq j \leq 2N
\end{array}$$

Functions φ and ψ are defined in (3),(4), (5), (6).

$$\begin{array}{l}
\varphi_{4j+1,k} = \gamma_k^{4j-1}\sin X_k, \quad \varphi_{4j+2,k} = \gamma_k^{4j}\cos X_k, \\
\varphi_{4j+3,k} = -\gamma_k^{4j+1}\sin X_k, \quad \varphi_{4j+4,k} = -\gamma_k^{4j+2}\cos X_k,
\end{array} \tag{3}$$

for $1 \leq k \leq N$, and

$$\begin{array}{l}
\varphi_{4j+1,N+k} = \gamma_k^{2N-4j-2}\cos X_{N+k}, \quad \varphi_{4j+2,N+k} = -\gamma_k^{2N-4j-3}\sin X_{N+k}, \\
\varphi_{4j+3,N+k} = -\gamma_k^{2N-4j-4}\cos X_{N+k}, \quad \varphi_{4j+4,N+k} = \gamma_k^{2N-4j-5}\sin X_{N+k},
\end{array} \tag{4}$$

for $1 \leq k \leq N$.

$$\begin{array}{l}
\psi_{4j+1,k} = \gamma_k^{4j-1}\sin Y_k, \quad \psi_{4j+2,k} = \gamma_k^{4j}\cos Y_k, \\
\psi_{4j+3,k} = -\gamma_k^{4j+1}\sin Y_k, \quad \psi_{4j+4,k} = -\gamma_k^{4j+2}\cos Y_k,
\end{array} \tag{5}$$

for $1 \leq k \leq N$, and

$$\begin{array}{l}
\psi_{4j+1,N+k} = \gamma_k^{2N-4j-2}\cos Y_{N+k}, \quad \psi_{4j+2,N+k} = -\gamma_k^{2N-4j-3}\sin Y_{N+k}, \\
\psi_{4j+3,N+k} = -\gamma_k^{2N-4j-4}\cos Y_{N+k}, \quad \psi_{4j+4,N+k} = \gamma_k^{2N-4j-5}\sin Y_{N+k},
\end{array} \tag{6}$$

for $1 \leq k \leq N$.
Arguments Y_k and Y_k are defined by

$$\begin{array}{l}
X_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{3,\nu}/2 - ie_\nu/2, \\
Y_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{1,\nu}/2 - ie_\nu/2,
\end{array}$$

for $1 \leq \nu \leq 2N$.
These terms are defined by means of λ_ν such that $-1 < \lambda_\nu < 1$, $\nu = 1, \ldots, 2N$,

$$\begin{array}{l}
-1 < \lambda_{N+1} < \lambda_{N+2} < \ldots < \lambda_{2N} < 0 < \lambda_N < \lambda_{N-1} < \ldots < \lambda_1 < 1 \\
\lambda_{N+j} = -\lambda_j, \quad j = 1, \ldots, N.
\end{array} \tag{7}$$

κ_ν, δ_ν and γ_ν are defined by

$$\begin{array}{l}
\kappa_j = 2\sqrt{1-\lambda_j^2}, \quad \delta_j = \kappa_j\lambda_j, \quad \gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}}, \\
\kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = 1/\gamma_j, \quad j = 1\ldots N.
\end{array} \tag{8}$$

The parameters a_j and b_j in the form

$$a_j = \sum_{k=1}^{N-1} \tilde{a_k} j^{2k+1}\epsilon^{2k+1}, \quad b_j = \sum_{k=1}^{N-1} \tilde{b_k} j^{2k+1}\epsilon^{2k+1}, \quad 1 \leq j \leq N. \tag{9}$$

Complex numbers e_ν $1 \leq \nu \leq 2N$ are defined by

$$e_j = ia_j - b_j, \quad e_{N+j} = ia_j + b_j, \quad 1 \leq j \leq N, \quad a, b \in \mathbf{R}. \tag{10}$$

The terms $x_{r,\nu}$ $(r = 3,\, 1)$ are defined by

$$x_{r,\nu} = (r-1) \ln \tfrac{\gamma_\nu - i}{\gamma_\nu + i}, \quad 1 \leq j \leq 2N. \tag{11}$$

Moreover, we have the following result which states that the highest amplitude of the modulus of the Peregrine breather of order N [1] :

Theorem 2.2. *The function v_0 defined by*

$$v_0(x,t) = \exp(2it - i\varphi) \times \left(\frac{\det((n_{jk})_{j,k \in [1,2N]})}{\det((d_{jk})_{j,k \in [1,2N]})} \right)_{(\tilde{a_j} = \tilde{b_j} = 0,\, 1 \leq j \leq N-1)} \tag{12}$$

is the Peregrine breather of order N solution to the NLS equation (1) whose highest amplitude in module is equal to $2N + 1$.

3 Hierarchy of solutions to NLS equation depending on $2N-2$ parameters

The solutions for orders $N = 3$ until $N = 10$ with $2N - 2$ parameters have been explicitly constructed by the present author [23, 24, 25, 26, 27, 28] with the help of M. Gastineau in the cases $N = 9$ and $N = 10$ [29, 30].
From these various studies, it appears that the solutions have quite particular structures depending on parameters $\tilde{a}_j$ and $\tilde{b}_j$. Parameters $\tilde{a}_j$ and $\tilde{b}_j$ play a similar role in obtaining the structures of the solutions. One can thus establish a certain number of conjectures about these solutions at the order N.
We illustrate these conjectures by figures of the solutions in the $(x;t)$ plane.

3.1 Case $a_1 \neq 0$ (or $b_1 \neq 0$)

For $\tilde{a}_1 \neq 0$ or $\tilde{b}_1 \neq 0$ and other parameters equal to 0, one obtains a triangle with $\frac{N(N+1)}{2}$ peaks.
It is important to note that a triangle is obtained only in this case; in all the other cases for only one parameter non equal to 0, we obtain rings.

[1] This result and the proof has been published in J.P.A [22]

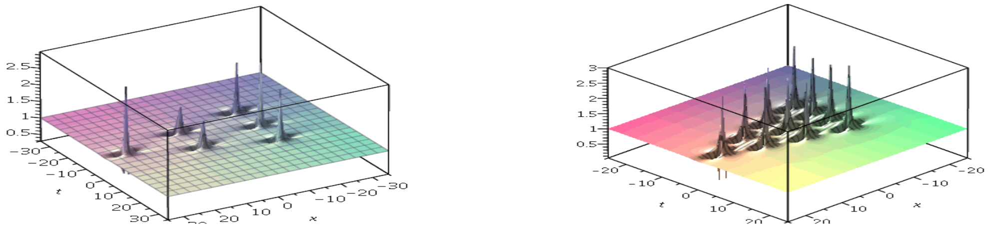

Figure 1: Solution to NLS, N=3, $\tilde{a}_1 = 10^4$: triangle with 6 peaks; on the right, N=4, $\tilde{a}_1 = 10^3$: triangle with 10 peaks.

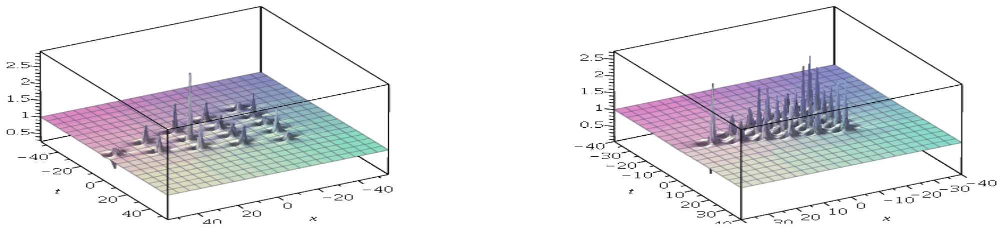

Figure 2: Solution to NLS, N=5, $\tilde{a}_1 = 10^4$: triangle with 15 peaks; on the right, N=6, $\tilde{a}_1 = 10^3$: triangle with 21 peaks.

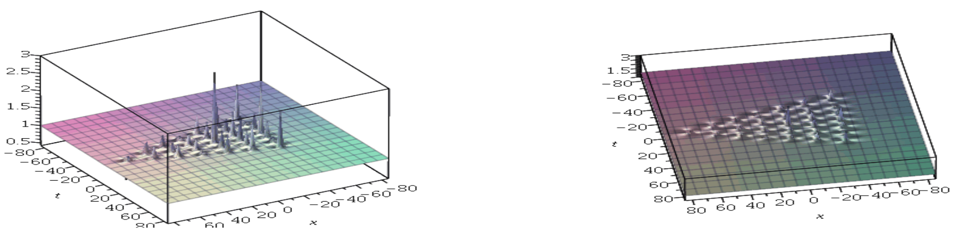

Figure 3: Solution to NLS, N=7, $\tilde{a}_1 = 10^4$: triangle with 28 peaks; on the right, sight from top.

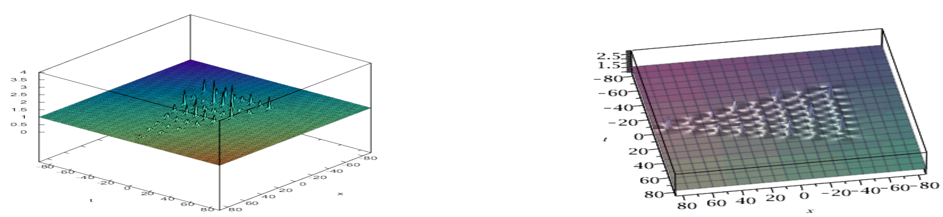

Figure 4: Solution to NLS, N=8, $\tilde{a}_1 = 10^6$: triangle with 36 peaks; on the right, sight from top.

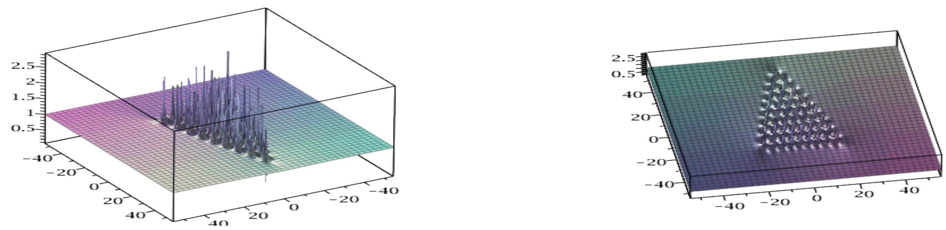

Figure 5: Solution to NLS, N=9, $\tilde{a}_1 = 10^3$: triangle with 45 peaks; on the right, sight from top.

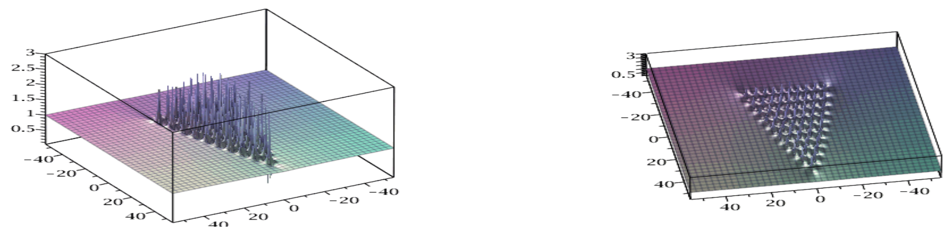

Figure 6: Solution to NLS, N=10, $\tilde{a}_1 = 10^3$: triangle with 55 peaks; on the right, sight from top.

3.2 Case $a_{N-1} \neq 0$ (or $b_1 \neq 0$), $N \geq 3$

For $\tilde{a}_{N-1} \neq 0$ or $\tilde{b}_{N-1} \neq 0$ and other parameters equal to 0, one obtains only one ring of $2N-1$ peaks with in the center Peregrine P_{N-2} of order $N-2$ [2]; here, $N \geq 3$.

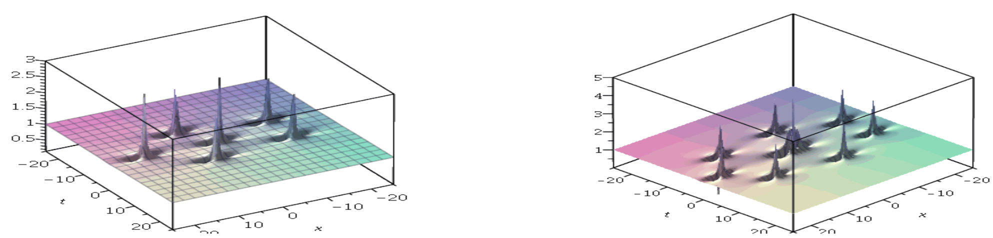

Figure 7: Solution to NLS, N=3, $\tilde{a}_2 = 10^6$: ring with 5 peaks, P_1 in the center; on the right, N=4, $\tilde{a}_3 = 10^8$: ring with 7 peaks, P_2 in the center.

[2]This conjecture has already been formulated by different authors, in particular by Akhmediev et al. [41]

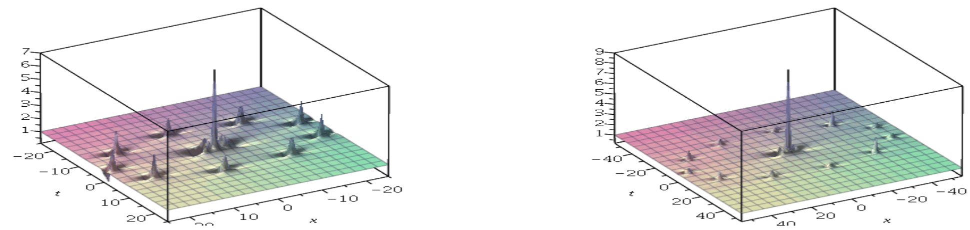

Figure 8: Solution to NLS, N=5, $\tilde{a}_4 = 10^{10}$: ring with 9 peaks, P_3 in the center; on the right, N=6, $\tilde{a}_5 = 10^{15}$: ring with 11 peaks, P_4 in the center.

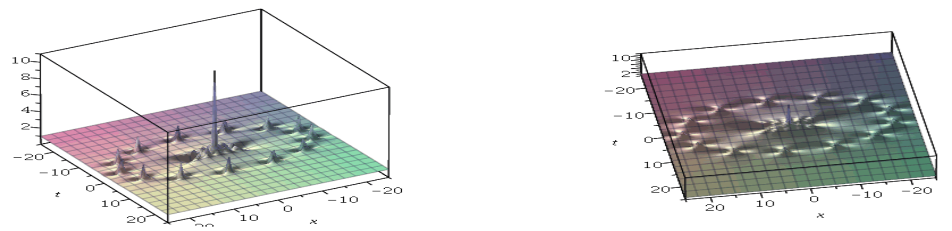

Figure 9: Solution to NLS, N=7, $\tilde{a}_6 = 10^{12}$: ring with 13 peaks, P_5 in the center; on the right, sight from top.

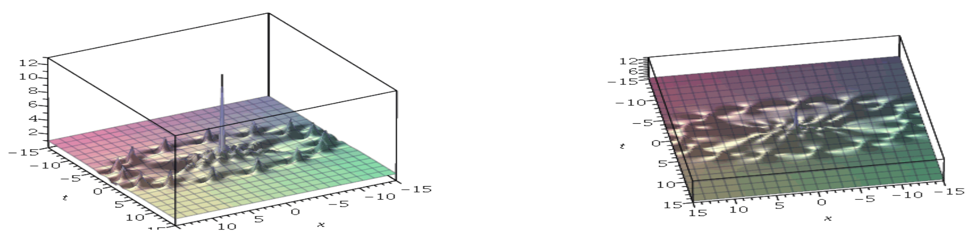

Figure 10: Solution to NLS, N=8, $\tilde{a}_7 = 10^{10}$: ring with 15 peaks, P_6 in the center; on the right, sight from top.

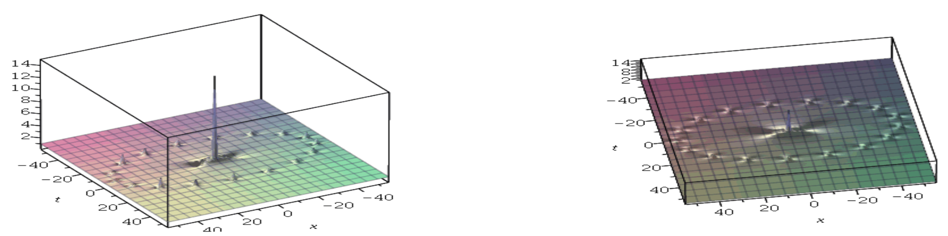

Figure 11: Solution to NLS, N=9, $\tilde{a}_8 = 10^{20}$: ring with 17 peaks, P_7 in the center; on the right, sight from top.

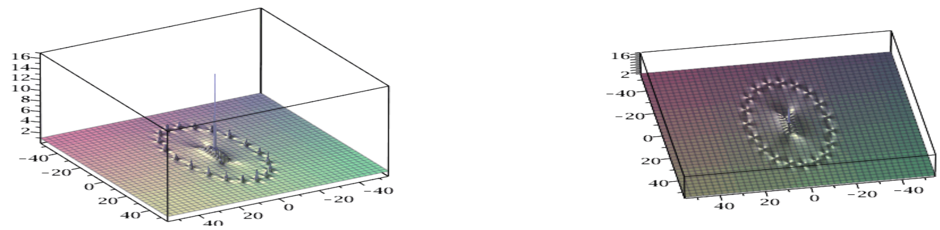

Figure 12: Solution to NLS, N=10, $\tilde{a}_9 = 10^{19}$: ring with 19 peaks, P_8 in the center; on the right, sight from top.

3.3 Case $a_{N-2} \neq 0$ (or $b_1 \neq 0$), $N \geq 5$

For $\tilde{a}_{N-2} \neq 0$ or $\tilde{b}_{N-2} \neq 0$ and other parameters equal to 0, one obtains two concentric rings of $2N-3$ peaks with in the center Peregrine P_{N-4} of order $N-4$; here $N \geq 5$.

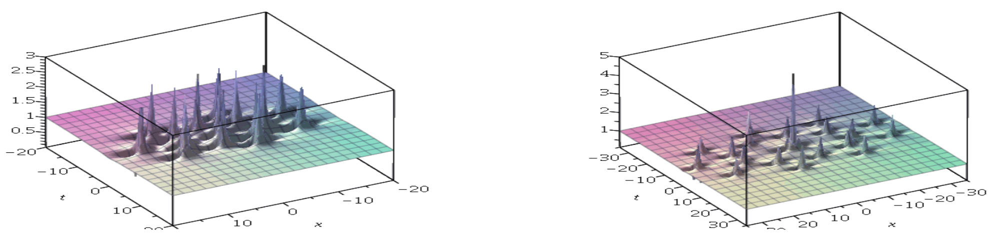

Figure 13: Solution to NLS, N=5, $\tilde{a}_3 = 10^6$: 2 rings with 7 peaks, P_1 in the center; on the right, N=6, $\tilde{a}_4 = 10^{10}$: 2 rings with 9 peaks, P_2 in the center.

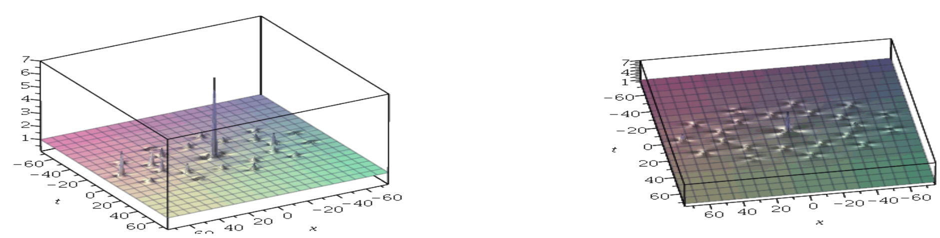

Figure 14: Solution to NLS, N=7, $\tilde{a}_5 = 10^{15}$: 2 rings with 11 peaks, P_3 in the center; on the right, sight from top.

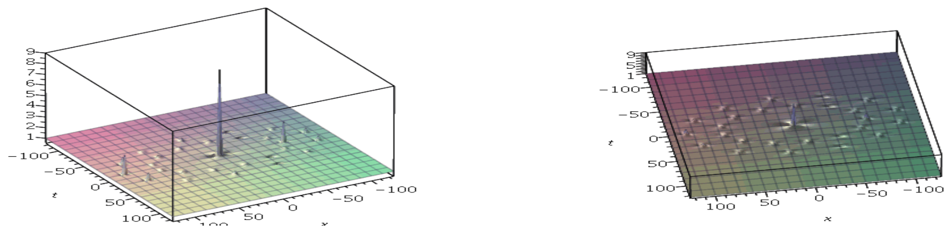

Figure 15: Solution to NLS, N=8, $\tilde{a}_6 = 10^{20}$: 2 rings with 13 peaks, P_4 in the center; on the right, sight from top.

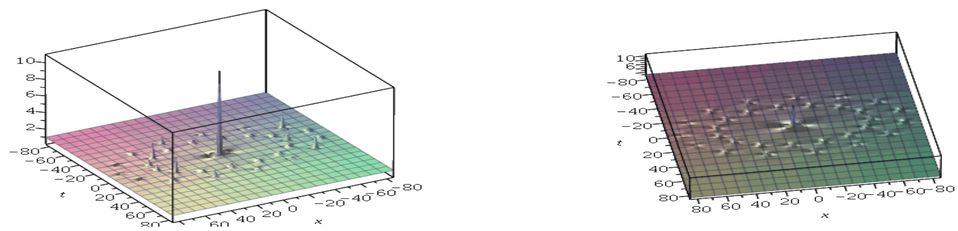

Figure 16: Solution to NLS, N=9, $\tilde{a}_7 = 10^{20}$: 2 rings with 15 peaks, P_5 in the center; on the right, sight from top.

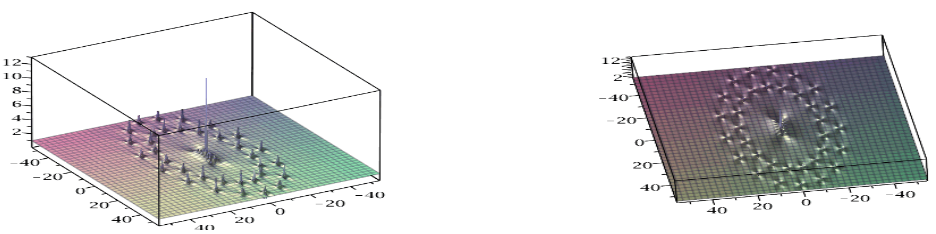

Figure 17: Solution to NLS, N=10, $\tilde{a}_8 = 10^{19}$: 2 rings with 17 peaks, P_6 in the center; on the right, sight from top.

3.4 Case $a_{N-3} \neq 0$ (or $b_{N-3} \neq 0$), $N \geq 7$

For $\tilde{a}_{N-3} \neq 0$ or $\tilde{b}_{N-3} \neq 0$ and other parameters equal to 0, one obtains three concentric rings of $2N-5$ peaks with in the center Peregrine P_{N-6} of order $N-6$; here $N \geq 7$.

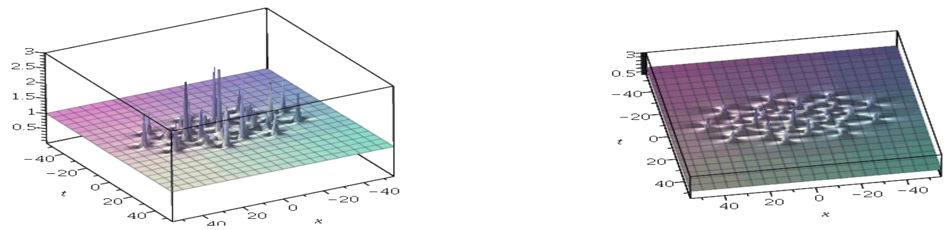

Figure 18: Solution to NLS, N=7, $\tilde{a}_4 = 10^{10}$: 3 rings with 9 peaks, P_1 in the center; on the right, sight from top.

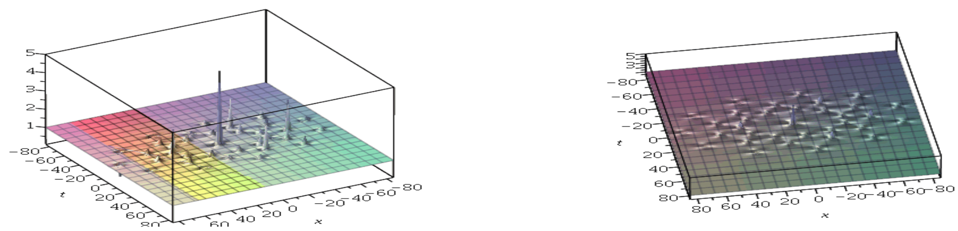

Figure 19: Solution to NLS, N=8, $\tilde{a}_5 = 10^{20}$: 3 rings with 11 peaks, P_2 in the center; on the right, sight from top.

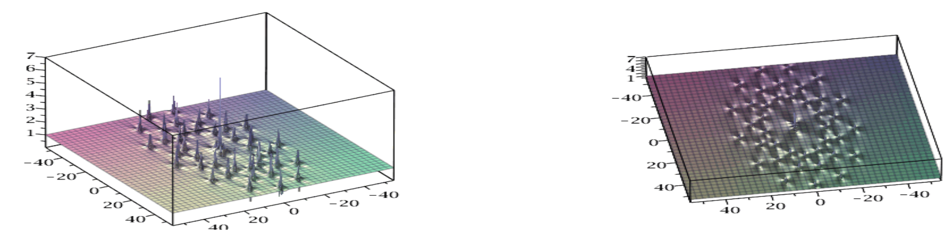

Figure 20: Solution to NLS, N=9, $\tilde{a}_6 = 10^{15}$: 3 rings with 13 peaks, P_3 in the center; on the right, sight from top.

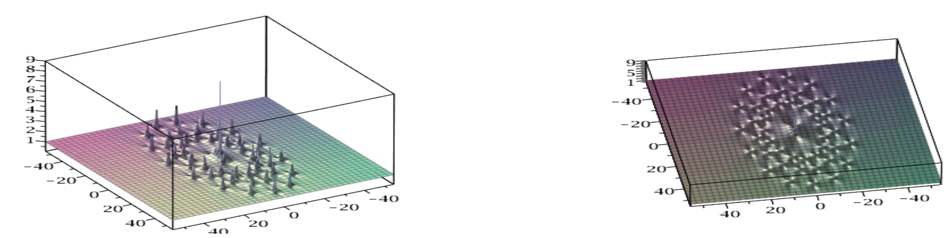

Figure 21: Solution to NLS, N=10, $\tilde{a}_7 = 10^{16}$: 3 rings with 15 peaks, P_4 in the center; on the right, sight from top.

3.5 General case

We recall the well known conjecture :
at order N, for $a_1 \neq 0$ or $b_1 \neq 0$, the modulus of the solution to the NLS equation presents the configuration of a triangle with $N(N+1)/2$ peaks in the (x,t) plane.

From the previous study, we can make the following conjecture :
at order N, for only one parameter non equal to 0, $a_{N-i} \neq 0$ or $b_{N-i} \neq 0$, the modulus of the solution to the NLS equation presents i concentric rings with $2N-2i+1$ peaks and in the center Peregrine breather of $N-2i$ order, for $1 \leq i \leq [\frac{N}{2}]$ in the (x,t) plane, with the convention that P_0 represent 0 peak.
It would be relevant to study the cases for the integers i such that $i > \frac{N}{2}$ and the parameters $a_{N-i} \neq 0$ or $b_{N-i} \neq 0$; the structure seems to be more complicated and would be clarified.

4 Conclusion

The structure of quasi-rational solutions to the one dimensional focusing NLS equation at order N has been given here as a product of an exponential depending on t by a ratio of two polynomials of degree $N(N+1)$ in x and t.
These solutions appear as $2N-2$-parameters deformations of the Peregrine breather P_N of order N : if $\tilde{a}_i = \tilde{b}_i = 0$ for $1 \leq i \leq N$, we obtain the classical Peregrine breather. This P_N breather has an higher amplitude in module equal to $2N+1$.
There are currently many applications in nolinear optics or hydrodynamics as recent works by Akhmediev et al. [42] or Kibler et al. [43] attest it in particular. A beginning of classification of the solutions to NLS equation was started with Akhmediev et al. [44]. It would be important in the future to prove the conjectures given in this paper and to give a complete classification for order N of the quasi rational solutions to the NLS equation.

Acknowledgments
I would like to thank M. Gastineau [45] for the data-processing assistance brought in the determination of the deformations in the cases $N = 9$ and $N = 10$ and for the realization of the corresponding images.
I would like also to thank M. Grundland for giving me the information about the classification of P. Winternitz.

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